SIMILARITY NUMBERS FOR A TURBULENT SHEAR FLOW OF A FLUID

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Certain relations in the semiempirical theory of turbulence are analyzed, for the purpose of generalizing the known transport models and deriving new functional relations.

Using the Karman method [1], we write the equation of flow

$$\rho \frac{D\mathbf{w}}{Dt} = -\operatorname{grad} p + \frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial \mathbf{w}}{\partial x_i} + \operatorname{grad} w_i\right)\right] - \frac{2}{3}\operatorname{grad}(\mu \operatorname{div} \mathbf{w}) \tag{1}$$

in a system of coordinates moving at an average velocity at any given point in the stream. The origin of a new system of coordinates will be placed at that particular point.

Considering a steady, uniform, and isothermal average flow along the z-axis, we obtain in the dimensionless coordinates

$$\alpha_i = x_i/l;\ f_i = w_i'/A, \tag{2}$$

the following equation

$$\begin{aligned}&\frac{\partial \mathbf{f}}{\partial t'} + f_i \frac{\partial \mathbf{f}}{\partial \alpha_i} + \frac{l}{A}\cdot\frac{dw}{dy}\left(\alpha_i \frac{\partial \mathbf{f}}{\partial \alpha_i} + f_i\right)\\ &= -\frac{l}{\rho A^2}\cdot\frac{d\bar{p}}{dz} - \frac{\operatorname{grad} p'}{\rho A^2} + \frac{\nu}{Al}\cdot\frac{\partial}{\partial \alpha_i}\left(\frac{\partial \mathbf{f}}{\partial \alpha_i} + \operatorname{grad} f_i\right)\\ &+ \frac{\nu l}{A^2}\cdot\frac{d^2 w}{dy^2} - \frac{2}{3}\cdot\frac{\nu}{Al}\operatorname{grad}\operatorname{div}\mathbf{f}.\end{aligned} \tag{3}$$

By the well known "Method of Equations" we can extract here four similarity numbers:

$$\frac{l}{A}\cdot\frac{dw}{dy}\ \text{(a)};\quad \frac{l}{\rho A^2}\cdot\frac{dp}{dz}\ \text{(b)};\quad \frac{Al}{\nu}\ \text{(c)};\quad \frac{l^2}{A}\cdot\frac{d^2w}{dy^2}\ \text{(d)}. \tag{4}$$

In its meaning, the (4c)-number corresponds to the Reynolds number for a pulsating flow.

According to the Third Theorem of Similarity, as the length scale l it will be convenient to choose either the distance from the wall or the dimensions of a displaced volume (mole) of fluid which produces velocity pulsations.

As a scale for the pulsation component of velocity it will be convenient to use its rms value. One may consider here two extreme cases: large and small values of the (4c)-number. Large values do, obviously, correspond to large-scale pulsations far from the wall. Small values correspond to small-scale pulsations either near the wall within the laminar sublayer or far from the wall but at small velocity pulsations.

We note that the (4c)-number appears in the denominator of the three last terms in Eq. (3), which account for the effect of viscosity.

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For this reason, in the extreme case of large-scale pulsations only the numbers (4a) and (4b) remain in Eq. (3), while in the case of small-scale pulsations there remain only the last three numbers: (4b), (4c), and (4d).

According to Newton's theorem, the similarity in the case of large-scale pulsations is thus characterized by a constant (4a)-number, which makes it possible to define a scale for the pulsation component of velocity:

$$A \sim l \frac{dw}{dy}. \tag{5}$$

This relation corresponds to the fundamental Prandtl expression in the theory of momentum transfer.

Consequently, the theory of momentum transfer is applicable only to large-scale pulsations far from the wall. Specifically, this explains the failure of attempts to approximate the velocity profile in the transition layer and in the laminar sublayer with the aid of relation (5). The similarity in the case of small-scale pulsations is characterized by a constant (4d)-number. Here a scale for the pulsation component of velocity is defined by the relation:

$$A \sim l^2 \frac{d^2w}{dy^2}. \tag{6}$$

Velocity pulsations within the laminar sublayer should vary according to this relation.

For intermediate values of the (4c)-number, the universal equation should contain all (4)-numbers. Specifically, multiplying (4a) by (4d) and equating their product to (4b), we obtain the well known relation which corresponds to Taylor's theory of vorticity transport:

$$\frac{dp}{dz} = \text{const}\, \rho l^2 \frac{dw}{dy} \cdot \frac{d^2w}{dy^2}. \tag{7}$$

If we divide (4a) by (4d), however, and consider the scales for the pulsation component of velocity to be proportional, then we obtain a relation which corresponds to Karman's "Similarity Theory of Velocity Pulsations":

$$l \sim \frac{dw}{dy} \Big/ \frac{d^2w}{dy^2}. \tag{8}$$

Consequently, Karman's "Similarity Theory" and Taylor's theory of vorticity transport correspond to the medium-scale flow region, where both large-scale and small-scale pulsations occur.

In the general case, according to the Second Law of Similarity, there are possible also other relations between the (4)-numbers, which will extend the applicability of these (4)-numbers beyond the limitations of the theories by Prandtl, Karman, and Taylor.

In particular, quite useful for calculating the velocity pulsations in the transition layer may be found the relation which involves the product of (4a) and (4d):

$$A^2 \sim l^3 \frac{dw}{dy} \cdot \frac{d^2w}{dy^2}. \tag{9}$$

Th. v. Karman [1] considered Eq. (1) without the three viscosity terms and thus omitted in his analysis important clues about the meaning and the role of the last two similarity numbers here: (4c) and (4d).

Relation (6), which Th. v. Karman has derived by a series expansion of the mean velocity, is rather random in nature, since it implies the possibility of infinitely many similarity numbers containing higher-order derivatives.

From the (4)-numbers one can derive an expression which extends the hypothesis concerning local properties to the case of molal and molecular transport [2].

Eliminating the scale for the pulsation component of velocity from (4a) and (4c), we obtain the expression

$$\frac{l^2}{\nu} \cdot \frac{dw}{dy} = R, \tag{10}$$

which is called in [2] the local Reynolds number. With the R-number as the independent variable, one can construct a smooth velocity profile either in parametric form [2] or in terms of a functional relation [3]. The group of (4)-numbers offers an interpretation of the stability criterion for a laminar sublayer in a gradient-type flow. The stability criterion is defined in [4] as

$$\frac{dw}{dy} \cdot \frac{y^2}{\nu} = 134. \tag{11}$$

It is to be noted that this relation is equivalent to (10) and reflects the application of the First Theorem of Similarity to the product of number (4a) by number (4c). Only large-scale pulsations have been considered here.

In addition to (11), one can also establish an analogous stability criterion for medium-scale pulsations:

$$\frac{d^2w}{dy^2} \cdot \frac{y^3}{\nu} = \text{const.} \tag{12}$$

In principle, it is possible to establish a stability criterion also on the basis of other combinations of the (4)-numbers.

In the case of variable thermophysical properties, we have instead of the (4d)-number:

$$\frac{l^2}{A\mu} \cdot \frac{d}{dy}\left(\mu \frac{dw}{dy}\right). \tag{13}$$

An analysis of the energy equation indicates the possibility of introducing the concept of large-scale and small-scale temperature pulsations, which are analogous in their meaning to respective velocity pulsations. Such an analysis has yielded two additional similarity numbers:

$$\frac{l}{B} \cdot \frac{dT}{dy} \text{ (a)}; \quad \frac{l^2}{B\lambda} \cdot \frac{d}{dy}\left(\lambda \frac{dT}{dy}\right) \text{ (b)}. \tag{14}$$

As an example of how these new relations between (4)-numbers can be applied, we will consider their effectiveness in determining the pulsation component of velocity according to (9).

Inserting into the well known original equation

$$\tau = \mu \frac{dw}{dy} + \psi\rho \sqrt{\overline{(w'_y)^2}} \sqrt{\overline{(w'_z)^2}} \tag{15}$$

the scales for the pulsation component of velocity from (9), and assuming a proportionality between longitudinal and transverse pulsations, and introducing the variables

$$\eta = \frac{y\sqrt{\dfrac{\tau}{\rho}}}{\nu}; \quad \varphi = \frac{w}{\sqrt{\dfrac{\tau}{\rho}}}, \tag{16}$$

we obtain

$$1 = \varphi' + f(\eta)\, \varphi'\varphi''\eta^3. \tag{17}$$

Here $f(\eta)$ denotes a proportionality factor which includes the correlation coefficient and which, generally, is a function of the distance from the wall. With respect to the first derivative here, Eq. (17) represents a special case of the Abel equation of the second kind and does not yield quadratures. We will solve this equation approximately by the iteration method.

Assuming, to the first approximation, that

$$\varphi' = 1/\chi\eta, \tag{18}$$

we find

$$|\varphi''| = 1/\chi\eta^2. \tag{19}$$

Inserting this value into (17), we obtain the second approximation:

$$\varphi' = \left(1 + f\,\frac{\eta}{\chi}\right)^{-1}. \tag{20}$$

The absolute value is taken in (19), because pulsations in both the positive and the negative direction produce positive shearing stresses (the Prandtl hypothesis).

The subsequent third approximation will be obtained by differentiating (20) and then inserting it into (17):

$$\varphi' = \left[1 + \frac{f'\eta + f}{\left(1 + f\,\frac{\eta}{\chi}\right)^2} \cdot \frac{f}{\chi}\,\eta^3\right]^{-1}. \tag{21}$$

Stopping at this approximation and assuming.

$$f(\eta) = c\sqrt{\eta},$$

we obtain an equation which corresponds to the velocity distribution in the case of the "fourth-power" law of pulsation penetration into the laminar sublayer:

$$\varphi' = \left[1 + \frac{\frac{3}{2}\,c^2\eta^4}{(1 + c\eta^{3/2}/\chi)^2\,\chi}\right]^{-1}. \tag{22}$$

Constants c and χ can be determined from the functional boundary conditions:

$$1)\ \text{when}\ \eta \to 0\ \ \varphi' = 1/(1 + (n\eta)^4), \tag{23}$$

with the Deisler constant n = 0.124, and

$$2)\ \text{when}\ \eta \gg 0\ \ \varphi' = 1/0{,}4\eta. \tag{24}$$

Applying the respective limits to Eq. (22) and then comparing with (23) and (24), we have

$$\frac{3}{2} \cdot \frac{c^2}{\chi} = (0.124)^4 \text{ and } \frac{3}{2}\,\varkappa = 0{,}4$$

or $\chi = 0.2667$ and $c^2 = 2/3\,(0.124)^4\,0.2667 = (0.0805)^4$. Inserting this value into (22) yields

$$\varphi' = \left\{1 + \frac{5.624\,(0.0805\eta)^4}{\left[1 + \frac{(0.0805\eta)^2}{0.2667\sqrt{\eta}}\right]^2}\right\}^{-1}. \tag{25}$$

A numerical integration of this expression yields a smooth velocity profile over the entire flow region which agrees closely with the experimental results obtained by H. Reihardt [5] and I. Laufer [6]. We note that the velocity profile based on (25) has been obtained by using only two empirical constants, while the other known profiles [5] and [7] involve at least three such constants.

If we let $f(\eta)$ = const., then Eq. (21) will yield a third-power law of pulsation penetration into the laminar sublayer, which closely approximates the velocity profile obtained experimentally even with only two constants.

The equation for the first derivative becomes then

$$\varphi' = \left[1 + \frac{2.5\,(0.103\eta)^3}{(1 + 0.805\eta)^2}\right]^{-1}. \tag{26}$$

An approximation of the velocity profile by a second-power or by a fifth-power law of pulsation penetration into the laminar sublayer has not been successful.

NOTATION

ρ is the density;
w is the velocity vector;
x_i is the Cartesian coordinate;
μ is the dynamic viscosity;

α_i	is the dimensionless coordinate;
l	is the lengths scale;
f_i	is the dimensionless pulsation component of velocity;
A	is the scale for the pulsation component of velocity;
p	is the average pressure;
p'	is the pulsation component of pressure;
ν	is the kinematic viscosity;
w	is the longitudinal component of average velocity;
z	is the longitudinal coordinate;
y	is the transverse coordinate;
T	is the temperature;
B	is the scale for the pulsation component of temperature;
τ	is the shearing stress;
ψ	is the correlation coefficient;
$\sqrt{\overline{(w'_y)^2}}, \sqrt{\overline{(w'_z)^2}}$	are the rms values of the pulsation component of velocity.

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